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The Orbits of all Groups of Order 24

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Abstract

Let G be a finite group of order 24 and $C = \{(a,b) \in G \times G : ab = ba, a \neq b, lcm(|a|,|b|) = 2\}$. In this paper, we provide the associated generalized conjugacy class graph for G via computing the probability of the set C which fixes by an element of G. Several properties of these graphs are given. We achieve these results with the aid of the GAP-Software (Groups, Algorithms and programming).

Introduction

The study of algebraic structures using the properties of graphs has become an exciting research topic in the last two decades, leading to many interesting results and questions. There are many papers on assigning a graph to groups, for example [1, 6, 9, 10]. For general notation and concepts in graph theory and group theory, we refer to [2] and [5], respectively. If C is a set and G is a group then G acts on C (C is G-set) if there is a function $\mathrm{G} \times \mathrm{C} \rightarrow \mathrm{C}$ denoted by $(\mathrm{g}, \mathrm{c}) \rightarrow \mathrm{g.c}$ such that $\mathrm{e.c} = \mathrm{c}$ and $(\mathrm{gh}).\mathrm{c} = \mathrm{g}.(\mathrm{h.c})$ for all g, h in G and c in C.

Let x and y be any two elements of a group G. Two elements x and y of G are called conjugate elements if there exists an element $\mathrm{z} \in \mathrm{G}$ such that $\mathrm{z} = \mathrm{x}^{-1}\mathrm{yx}$. The set of all conjugate elements of y by an element of G is called the conjugacy class of y and denoted by $\mathrm{cl(y)}$. Its size is computed by the Orbit-Stabilizer Theorem. It is remarkable to mention that if x is an element of the center of a group then its conjugacy class is $\{\mathrm{x}\}$. The concept of conjugate graphs of finite groups, which is introduced by Erfanian and Tolue in [3], is based on the conjugation of two non-central elements of a group. They assign a graph to a finite nonabelian group G with vertex set of noncentral elements of the group such that two distinct vertices are connected if and only if they are conjugate. In addition, the authors in [9] studied the behavior of orbits of dihedral groups of order 16, 18 and 20 as they constructed the generalized conjugacy class graph. Their work is based on group action on a specific set of the group that their study is based on [1]. Orbits have significant role in both group and graph theory that are be used to find some properties of the given groups and graphs.

In this paper, the generalized conjugacy class graph is formulated to demonstrate the behavior of the G-orbits and the probability of fixing the set C by an element of a group is calculated for given groups. The vertex set is the non-central orbits under G-action on the set C and two vertices are connected if the cardinalities of the orbits are not coprime. In [10], the authors found the generalized conjugacy class graphs and their properties for three non-abelian groups of order 24. However, the rest of all groups of order 24 are classified in this paper.

Preliminaries

In this section, some basic concepts and properties of graph and group theory are given. Let G be a non-abelian group of order 24 and

$$C = \{(a,b) \in G \times G : ab = ba, a \neq b, lcm(|a|,|b|) = 2\} \quad (1)$$

Definition 2. [7]

Let G be a finite group and C be a G-set. The vertices of generalized conjugacy class graph are $K(C) - |A|$, where $K(C)$ is the number of non-central conjugacy classes under group action on C and $A = \{c \in C, cg = gc, g \in G\}$. Let x, y be two vertices which are connected by an edge if their cardinalities are not co-prime. It is denoted by Γ_G^C.

Definition 3. [5]

Suppose G acts on a set C and $\in C$. The orbit of c, $O(c)$, is the subset

$O(c) = \{g * c \mid g \in G, c \in C\}$.

In this paper, we assume that G acts on itself by conjugation, therefore, the orbit can be written as $(c) = \{c^g \mid g \in G\}$.

Definition 4. [7]

Let G be a finite group and G acts on C by conjugation. Then the probability that an element of a group fixes a set C is given by:

$P_G(C) = \frac{|O|}{|C|}$

where $|O|$ is the number of conjugacy classes of C in G.

Definition 6. [2]

A graph is called a complete graph K_n(with n vertices), if all of the vertices in a graph are adjacent to each other.

Definition 7. [2]

A graph with no vertices is called the null graph.

Definition 8. [2]

A graph Γ is n-colourable if it has a n-coloring. The minimum n for which a graph Γ is n-colourable is called its chromatic number, and denoted by $\chi(\Gamma)$.

Definition 9. [2]

A clique of a graph Γ is a set of mutually adjacent vertices, and that the maximum size of a clique of Γ is called clique number of Γ and denoted by $\omega(\Gamma)$.

Algorithm

There are 15 small groups of order 24. Three of them are abelian groups which give the null graphs and probabilities one. The others are not abelian. The following steps with some GAP-codes are used to obtain the following results.

Step1 The list of small groups can be obtained from the GAP codes:

```
h:=AllSmallGroups(24);
```

To pick the non-abelian groups list, The GAP codes are used:

```
h:=AllSmallGroups(Size,24,IsAbelain,false);
```

Step2 Computing the set C by using The GAP codes:

$a * b = b * a \; and \; LcmInt(Order(a), Order(b)) = 2$

Step3 Computing the number of G-orbits on C and its probability.
Step4 To find the conjugacy class graph: by computing central orbits and subtracted from the number of G-orbits gives the numbers of vertices of the graph. Two verteices are joined by an edge if their cardinalities are not coprime.

For instance, the presentations of the groups S_4 and D_{24} are given as follows:

$$S_4 = < a, b \mid a^3, b^2, (ab)^4 >$$

and

$$D_{24} = < a, b \mid a^{12}, b^2, b^{-1} \ a\ b\ a >$$

because they are used in the following results.

The following result can be found in [10].

Proposition 1.

Let G be the Dihedral group D_{24}. Then the probability of fixing the set C by an element of G is $P_G(C) = \frac{6}{31}$. Furthermore $\Gamma_G^C = K_{10}$.

Main Results

In this section, the generalizations conjugacy class graph of all non-abelian groups of order 24 are presented and several properties of it are given. Throughout this section we assume that G acts on C be conjugation.

Proposition 2.

Let G be the symmetric group S_4. Then the probability of fixing the set C by an element of G is $P_G(C) = \frac{4}{21}$. Furthermore $\Gamma_G^C = K_8$.

Proof.

The conditions which are exist in C, give 42 elements of the set C, which are the following:

$C = \{(e, a^2ba), (e, ba(ab)^2), (e, b), (e, (a^2b)^2ab), (e, (ab)^2),$
$(e, (aba)^2b), (e, (aba)^2), (e, aba^2), (e, (a^2b)^2), (a^2ba, e),$
$(a^2ba, (a^2b)^2ab), (a^2ba, (ab)^2), (ba(ab)^2, e), (ba(ab)^2, aba^2)$
$, (ba(ab)^2, (a^2b)^2), (b, e), (b, (aba)^2b), (b, (aba)^2), ((a^2b)^2ab, e)$
$, ((a^2b)^2ab, a^2ba), ((a^2b)^2ab, (ab)^2), ((ab)^2, e), ((ab)^2, a^2ba),$
$((ab)^2, (a^2b)^2ab), ((ab)^2, (aba)^2), ((ab)^2, (a^2b)^2), ((aba)^2b, e),$
$((aba)^2b, b), ((aba)^2b, (aba)^2), ((aba)^2, e), ((aba)^2, b),$
$((aba)^2, (ab)^2), ((aba)^2, (aba)^2b), ((aba)^2, (a^2b)^2) , (aba^2, e)$
$, (aba^2, ba(ab)^2), (aba^2, (a^2b)^2), ((a^2b)^2, e), ((a^2b)^2, ba(ab)^2),$
$((a^2b)^2, (ab)^2), ((a^2b)^2, (aba)^2), ((a^2b)^2, aba^2)\}$.

Since S_4 acts on C by conjugation, there are 8 orbits:

1. $O_1 = \{(e, a^2ba), (e, ba(ab)^2), (e, b), (e, (a^2b)^2ab), (e, (aba)^2b), (e, aba^2)\}$
2. $O_2 = \{(e, (ab)^2), (e, (aba)^2), (e, (a^2b)^2)\}$
3. $O_3 = \{(a^2ba, e), (ba(ab)^2, e), (b, e), ((a^2b)^2ab, e), ((aba)^2b, e), (aba^2, e)\}$
4. $O_4 = \{(a^2ba, (a^2b)^2ab), (ba(ab)^2, aba^2), (b, (aba)^2b), ((a^2b)^2ab, a^2ba), ((aba)^2b, b), (aba^2, ba(ab)^2)\}$
5. $O_5 = \{(a^2ba, (ab)^2), (ba(ab)^2, (a^2b)^2), (b, (aba)^2), ((a^2b)^2ab, (ab)^2), ((aba)^2b, (aba)^2), (aba^2, (a^2b)^2) \}$
6. $O_6 = \{((ab)^2, e), ((aba)^2, e), ((a^2b)^2, e)\}$

7. $O_7 = \{((ab)^2, a^2ba), ((ab)^2, (a^2b)^2ab), ((aba)^2, b),$
$((aba)^2, (aba)^2b), ((a^2b)^2, ba(ab)^2), ((a^2b)^2, aba^2)\}$
8. $O_8 = \{((ab)^2, (aba)^2), ((ab)^2, (a2b)^2), ((aba)^2, (ab)^2)$
$, ((aba)^2, (a^2b)^2), ((a^2b)^2, (ab)^2), ((a^2b)^2, (aba)^2)\}$.

From Definition 4 the probability of fixing the set C by an element of S_4 is $P_{S_4}(C) = \frac{4}{21}$. Thus S_4 has 8 orbits, which are all non-central orbits, that is $A = \emptyset$. Moreover, since the greatest common multiple of the cardinalities of all non-central orbits are not one then the generalized conjugacy class graph is the complete graph K_8.

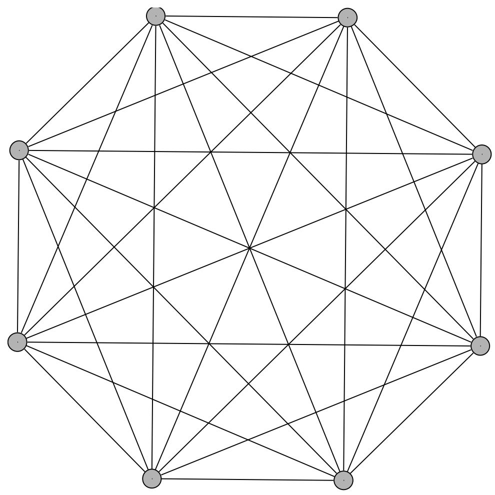

Figure 1: Generalized Conjugacy Class Graph of S_4

Proposition 3.
If G is one of the groups $C_3 \rtimes C_8$, $SL(2,3)$, $C_3 \rtimes Q_8$ and $C_3 \times Q_8$, then the probability of fixing the set C by an element of G is $P_G(C) = 1$. Furthermore $\Gamma_G^C = \emptyset$.

Proof.
If G is any one of $SL(2,3)$ or $C_3 \times Q_8$. The group G has only one element of order two, which is a. From Definition 2, we have $C = \{(e, a^2), (a^2, e)\}$. Hence, the order of C is two.
Since the G-action on C is conjugation, then the orbits are
1. $O_1 = \{(e, a^2)\}$.
2. $O_2 = \{(a^2, e)\}$.
Thus, there are two orbits and $P_G(C) = 1$.
Finally, The proofs for the groups $C_3 \rtimes C_8$, and $C_3 \rtimes Q_8$, can be found in [10].

Proposition 4.
Let G be the group $C_2 \times (C_3 \rtimes C_4)$. Then the probability of fixing the set C by an element of G is $P_G(C) = 1$. Furthermore $\Gamma_G^C = \emptyset$.
Proof.
The proof is similar to Proposition 2.

Proposition 5.
Let G be the group $C_4 \times S_3$. Then the probability of fixing the set C by an element of G is
$P_G(C) = \frac{3}{8}$. Furthermore $\Gamma_G^C = K_{10}$.
Proof.

The proof is similar to Proposition 2.

Proposition 6.

Let G be the group $(C_2 \times C_2) \rtimes C_2$. Then the probability of fixing the set C by an element of G is $P_G(C) = \frac{2}{7}$. Furthermore $\Gamma_G^C = K_{10}$.

Proof.

The proof is similar to Proposition 2.

Proposition 7.

Let G be the group $C_3 \times D_8$. Then the probability that an element of G fixes the set C is $P_G(C) = \frac{6}{11}$. Furthermore $\Gamma_G^C = K_{10}$.

Proof.

The proof is similar to Proposition 2.

Proposition 8.

Let G be the group $C_2 \times A_4$. Then the probability of fixing the set C by an element of G is $P_G(C) = \frac{5}{14}$. Furthermore $\Gamma_G^C = K_{18}$.

Proof.

The proof is similar to Proposition 2.

Proposition 9.

Let G be the group $C_2 \times C_2 \times S_3$. Then probability of fixing the set C by an element of G is $P_G(C) = \frac{7}{18}$. Furthermore $\Gamma_G^C = K_{44}$.

Proof.

The proof is similar to Proposition 2.

From the results in [10] and the above propositions, the chromatic and clique numbers can be found as their graphs are complete or null graph.

Conclusion

The results are summarized in Table 1, where |C| is the cardinality of the set C which is defined in (1), $\#O$ is the number of orbits obtained from the G-action on C, P is the probability of fixing the set C by an element of G and $|A|$ is the number of commuting element from definition 3 and Γ_G^C is the generalized conjugacy class graph.

Table 1: Non-Abelian Groups of Order 24

Group	$\|C\|$	$\#O$	P	$\|A\|$	Γ_G^C	$\chi(\Gamma_G^C)$	$\omega(\Gamma_G^C)$
$C_3 \rtimes C_8$	2	2	1	2	Null graph	0	0
$SL(2,3)$	2	2	1	2	Null graph	0	0
$C_3 \rtimes Q_8$	2	2	1	2	Null graph	0	0
$C_4 \times S_3$	32	12	$\frac{3}{8}$	2	K_{10}	10	10
D_{24}	62	12	$\frac{6}{31}$	2	K_{10}	10	10
$C_2 \times (C_3 \rtimes C_4)$	12	12	1	12	Null graph	0	0
$(C_6 \times C_2) \rtimes C_2$	42	12	$\frac{2}{7}$	2	K_{10}	10	10

$C_3 \times D_8$	22	12	$\frac{6}{11}$	2	K_{10}	10	10
$C_3 \times Q_8$	2	2	1	2	Null Graph	0	0
S_4	42	8	$\frac{4}{21}$	0	K_8	8	8
$C_2 \times A_4$	56	20	$\frac{5}{14}$	2	K_{18}	18	18
$C_2 \times C_2 \times S_3$	144	56	$\frac{7}{18}$	12	K_{44}	44	44

At the end, we classified these groups by the number of orbits into three types which are:

1. Groups with no non-central orbits.
2. Groups with the same number of non-central orbits.
3. Others with different numbers of non-central orbits.

It is noticed that there are five groups that do not have noncentral conjugacy class, four groups have the same number of conjugacy classes and other three groups have different numbers of conjugacy classes.

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